The Open University

MST121
Using Mathematics

Exercise Book A

Contents

The exercises in this booklet are intended to give further practice, should you require it, in handling the main mathematical ideas in each chapter of MST121, Block A. The exercises are ordered by chapter and section, and are numbered correspondingly: for example, Exercise 3.2 for Chapter A1 is the second exercise on Section 3 of that chapter.

Exercises for Chapter A1

Section 1

Exercise 1.1

For each of the following sequences, write down the first five terms and the 10th term of the sequence, and plot a graph showing the first five terms. Give your answers as fractions for parts (a) and (b), and to 3 decimal places for part (c).

(a) $a_n = n^2 - 3n \quad (n = 1, 2, 3, \ldots)$

(b) $b_n = \dfrac{2}{n} \quad (n = 1, 2, 3, \ldots)$

(c) $c_n = n^{1/3} \quad (n = 0, 1, 2, \ldots)$

Exercise 1.2

Write down the first five terms of each of the following sequences, giving your answers as fractions.

(a) $a_1 = 0, \quad a_{n+1} = a_n^2 + 1 \quad (n = 1, 2, 3, \ldots)$

(b) $b_1 = 1, \quad b_{n+1} = b_n^2 - 1 \quad (n = 1, 2, 3, \ldots)$

(c) $c_0 = 0, \quad c_{n+1} = \dfrac{1}{1 + c_n} \quad (n = 0, 1, 2, \ldots)$

Section 2

Exercise 2.1

Each of the following sequences is arithmetic and infinite. For each sequence,

(i) find the values of the parameters a and d, and write down the corresponding recurrence system;

(ii) calculate the next two terms, and plot a graph of the first six terms.

(a) The sequence x_n whose first four terms are 17, 21, 25, 29.

(b) The sequence y_n whose first five terms are 4.4, 4.1, 3.8, 3.5, 3.2.

Exercise 2.2

Write down a closed form for each of the following sequences, and hence calculate the 12th term of each sequence.

(a) $x_1 = -7, \quad x_{n+1} = x_n + 6 \quad (n = 1, 2, 3, \ldots)$

(b) $y_0 = 9.1, \quad y_{n+1} = y_n - 0.8 \quad (n = 0, 1, 2, \ldots)$

Section 3

Exercise 3.1

Each of the following sequences is geometric and infinite. For each sequence,

(i) find the values of the parameters a and r, and write down the corresponding recurrence system;

(ii) calculate the next two terms, giving your answers rounded to 3 decimal places, and plot a graph of the terms you know.

(a) The sequence x_n whose first three terms are 2, 2.8, 3.92.

(b) The sequence y_n whose first four terms are 64, −48, 36, −27.

Exercise 3.2

Write down a closed form for each of the following sequences, and hence calculate the 12th term of each sequence, correct to 3 significant figures.

(a) $x_1 = -3, \quad x_{n+1} = -2.1x_n \quad (n = 1, 2, 3, \ldots)$

(b) $y_0 = 83, \quad y_{n+1} = 0.79y_n \quad (n = 0, 1, 2, \ldots)$

Section 4

Exercise 4.1

Each of the following is an infinite linear recurrence sequence. For each sequence,

(i) find the values of the parameters a, r and d, and write down the corresponding recurrence system;

(ii) calculate the next two terms, giving your answers correct to 3 decimal places, and plot a graph of the terms you know.

(a) The sequence x_n whose first three terms are 5, 10, 9.

(b) The sequence y_n whose first four terms are 3, 2.9, 2.67, 2.141.

Exercise 4.2

Write down a closed form for each of the following sequences, and hence calculate the 9th term of each sequence, giving your answer correct to 4 significant figures.

(a) $x_1 = 2, \quad x_{n+1} = 4x_n + 3 \quad (n = 1, 2, 3, \ldots)$

(b) $y_0 = 7, \quad y_{n+1} = -\tfrac{1}{3}y_n - 1 \quad (n = 0, 1, 2, \ldots)$

(c) $z_1 = 2, \quad z_{n+1} = 2.5z_n \quad (n = 1, 2, 3, \ldots)$

Exercise 4.3

Each of the following is an infinite linear recurrence sequence. Decide which of them, if any, are arithmetic and which are geometric. (If you would like the extra practice, then you could also find the corresponding recurrence system and closed form, which we give in the solutions.)

(a) The sequence a_n whose first four terms are 8.5, 6.8, 5.44, 4.352.

(b) The sequence b_n whose first four terms are 8.5, 6.8, 5.1, 3.4.

(c) The sequence c_n whose first four terms are 8, −6, 11.5, −10.375.

(d) The sequence d_n whose first four terms are 8, −6, 4.5, −3.375.

Section 5

Exercise 5.1

Describe the long-term behaviour of each of the following sequences.

(a) $a_n = \frac{1}{100}(-1.09)^n - 919 \quad (n = 0, 1, 2, \ldots)$

(b) $b_n = 0.007n - 3013 \quad (n = 1, 2, 3, \ldots)$

(c) $c_n = 872(0.41)^{n-1} + 1.99 \quad (n = 1, 2, 3, \ldots)$

Exercise 5.2

Describe the long-term behaviour of each of the following sequences.

(a) The sequence x_n given by
$x_1 = -5, \quad x_{n+1} = 0.4x_n + 12 \quad (n = 1, 2, 3, \ldots).$

(b) The sequence y_n given by
$y_0 = 2, \quad y_{n+1} = -0.8y_n - 27 \quad (n = 0, 1, 2, \ldots).$

Exercise 5.3

Describe the long-term behaviour of each of the following, all of which are infinite linear recurrence sequences.

(a) The sequence a_n whose first four terms are −6, −5.8, −5.6, −5.4.

(b) The sequence b_n whose first four terms are −1200, −1080, −972, −874.8.

(c) The sequence c_n whose first four terms are 36, 18, 54, −18.

Exercises for Chapter A2

Section 1

Exercise 1.1

Sketch the line which corresponds to each of the following equations.

(a) $y = -\frac{1}{4}x$

(b) $y = 3x - 1$

(c) $y = -2x - 1$

Exercise 1.2

Find the x-intercept and y-intercept for the lines given by each of the following equations.

(a) $y = -\frac{4}{3}x + 2$

(b) $y = \frac{1}{3}x$

Exercise 1.3

Find the equations of each of the following lines.

(a) The line with slope 4 which passes through the point $(-3, 2)$.

(b) The line which passes through the points $(-3, 2)$ and $(1, -4)$.

(c) The line which passes through the points $(-3, 2)$ and $(7, 2)$.

(d) The line which has x-intercept −3 and y-intercept 3.

Exercise 1.4

(a) The point A has coordinates $(-1, -2)$ and the point B has coordinates $(3, 5)$. Find the slope of the line through A and B.

(b) What is the slope of each line perpendicular to AB?

(c) Find the equation of the line which passes through A and is perpendicular to AB.

Exercise 1.5

Find the equation of the line which passes through the point $(-1, 1)$ and is perpendicular to the line $y = -\frac{3}{5}x + \frac{2}{5}$.

Exercise 1.6

For each of the following pairs of lines, find their point of intersection.

(a) $y = -2x + 1$ and $y = 7x - 2$

(b) $x = 3$ and $y = 7x - 2$

Section 2

Exercise 2.1

Find the distance between each of the following pairs of points.

(a) $(8, 9)$ and $(2, 7)$ (b) $(-5, 0)$ and $(-2, -3)$

Exercise 2.2

Write down the equation of the circle in the (x, y)-plane which satisfies each of the following specifications.

(a) Centre at $(0, 0)$ and radius 6.

(b) Centre at $(-2, 0)$ and radius $\frac{1}{3}$.

(c) Centre at $(4, -1)$ and radius $\sqrt{3}$.

Exercise 2.3

Write down the centre and radius of the circle specified by each of the following equations.

(a) $(x - 4)^2 + (y - 2)^2 = 9$

(b) $(x + 5)^2 + (y - 3)^2 = 64$

(c) $(x - \sqrt{2})^2 + y^2 = 8$

Exercise 2.4

This exercise concerns the three points $A(-1, 3)$, $B(1, -7)$ and $C(5, 13)$.

(a) Find the slope of the line segment AB, and the coordinates of the midpoint M of AB.

(b) Find the equation of the line through M perpendicular to AB.

(c) Find the equation of the perpendicular bisector of the line segment BC.

(d) Find the point D where the lines in parts (b) and (c) intersect.

(e) Hence find the equation of the circle through the points A, B and C.

Exercise 2.5

Find the centre and radius of the circle which passes through the three points $A(-3, 0)$, $B(5, 4)$ and $C(2, 10)$. Write down the equation of this circle.

Exercise 2.6

Find the completed-square form of each of the following expressions.

(a) $x^2 + 20x$

(b) $x^2 - 11x$

(c) $y^2 + 7y$

Exercise 2.7

For each of the following equations, verify that it represents a circle, and find its centre and radius.

(a) $x^2 + 12x + y^2 - 6y + 40 = 0$

(b) $x^2 - 5x + y^2 + 3y - \frac{1}{2} = 0$

Exercise 2.8

Find any points at which the circle $(x + 3)^2 + (y - 4)^2 = 29$ intersects each of the following lines.

(a) $y = -2x - 1$ (b) $y = 2x - 3$

Section 3

Exercise 3.1

(a) Express the following angles, measured in degrees, in terms of radians.

(i) $720°$ (ii) $135°$ (iii) $210°$

(b) Express the following angles, measured in radians, in terms of degrees.

(i) 3π (ii) $\frac{4}{3}\pi$ (iii) $\frac{7}{4}\pi$

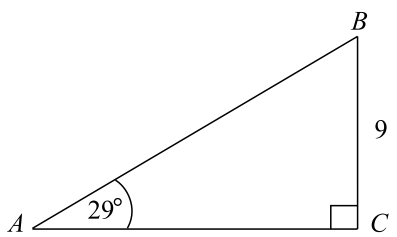
Exercise 3.2

Write down the sine, cosine and tangent of each of the angles in Exercise 3.1 above.

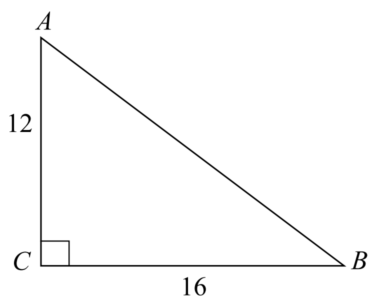
Exercise 3.3

Find the unmarked angles and side lengths for each of the following three right-angled triangles.

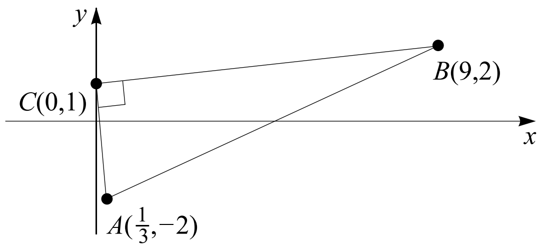
(a)



(b)

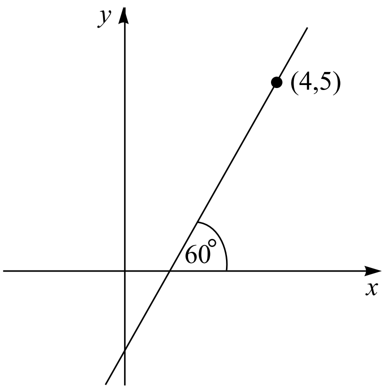


(c)



Exercise 3.4

Find the equation of the line through the point $(4, 5)$ which makes an angle of $60°$ with the positive x-axis, when measured anticlockwise.



Section 4

Exercise 4.1

Write down parametric equations for:

(a) the line which has slope 4 and passes through the point $(-3, 2)$;

(b) the line which passes through the two points $(-1, 3)$ and $(2, -2)$;

(c) the line segment joining the points $(-6, 5)$ and $(-2, -1)$, including an appropriate range for the parameter t.

Exercise 4.2

From the parametric equations

$$x = 2t - 7, \quad y = -6t + 3,$$

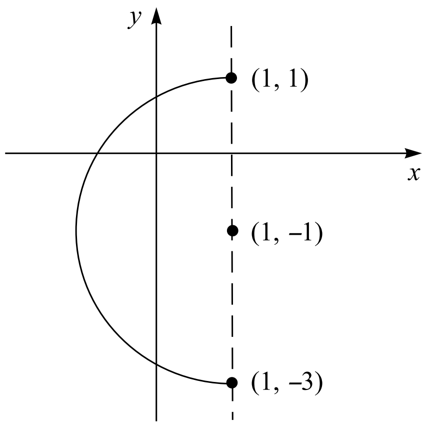
deduce the equation of the corresponding line.

Exercise 4.3

Write down parametric equations, including an appropriate range for the parameter t, for each of the following.

(a) One revolution of the circle with centre $(-1, 5)$ and radius 4, starting and finishing at $(3, 5)$.

(b) The semicircle below.



Exercises for Chapter A3

Section 1

Exercise 1.1

For each of the following functions f, write down a specification of the function, and find the image of the number -1 under f.

(a) The function f which converts any negative real number x into its cube.

(b) The function f which converts any real number x which satisfies $-2 \le x \le 0$ into $2 - x^2$.

Exercise 1.2

For each of the following inequalities, write down the corresponding interval, and describe it as closed, open or half-open (half-closed).

(a) $-3 \le x < -2$

(b) $-3 \le x$

(c) $-3 < x < 2$

(d) $x < 2$

Exercise 1.3

Which of the following intervals include the number -2?

$$(-5,0),\quad [-3,-2),\quad (-1,3),\quad (-3,-2],\quad (-2,\infty).$$

Exercise 1.4

For each of the following functions, write down the domain of the function using interval notation.

(a) $f(x) = x^2 + 3 \quad (x \geq 0)$

(b) $f(x) = (\sqrt{x-1}\,)^3$

(c) $f(x) = \dfrac{x}{x+1} + \dfrac{3}{\sqrt{x+2}}$

Section 2

Exercise 2.1

A circular pond of radius r metres is placed at the centre of a 10 metre by 15 metre rectangular plot and the remainder of the plot is sown with grass seed to give a lawn of area A square metres.

(a) Write down the largest closed interval consisting of values of r which make sense for this problem.

(b) Express A in terms of r.

(c) Find the value of r for which A is two-thirds of the area of the whole plot.

Exercise 2.2

Rearrange each of the following expressions in completed-square form.

(a) $x^2 - 14x + 30$

(b) $3x^2 + 5x + 1$

(c) $-4x^2 + x + 2$

Exercise 2.3

Sketch the graphs of each of the following functions by using translations and scalings of the graph of $y = x^2$, and by finding the x- and y-intercepts (if any).

(a) $f(x) = 3x^2 + 18x + 15$

(b) $f(x) = -3x^2 + 2x - \frac{7}{3}$

Exercise 2.4

Find the image set of each of the following functions. (Exercise 2.3(a) above will be of help.)

(a) $f(x) = 3x^2 + 18x + 15 \quad (-2 \leq x \leq 0)$

(b) $f(x) = 3x^2 + 18x + 15 \quad (-4 \leq x \leq -1)$

Exercise 2.5

Find the solutions, if any, of each of the following equations.

(a) $|3x + 7| - 4 = 0$

(b) $|3x - 7| + 4 = 0$

Exercise 2.6

Sketch the graph of the function

$$f(x) = \tfrac{1}{2}|x+2| - \tfrac{3}{2},$$

by using translations and scalings of the graph of $y = |x|$, and by finding the x- and y-intercepts (if any).

Section 3

Exercise 3.1

By considering the effect of appropriate translations on the graph of the function $f(x) = e^x$, sketch the graph of the function $g(x) = e^{x-1} + 1$.

Exercise 3.2

By considering the effect of appropriate scalings on known graphs, sketch each of the following graphs.

(a) $y = e^{x/2}$

(b) $y = \sin(\pi x)$

Exercise 3.3

By an appropriate adjustment of the graph of the sine function, sketch the following graphs. (Use values of x in the interval $[-2\pi, 2\pi]$.)

(a) $y = \sin|x|$

(b) $y = |\sin x|$

Section 4

Exercise 4.1

For each of the following functions f, find the inverse function f^{-1}, and sketch the graph of f^{-1}.

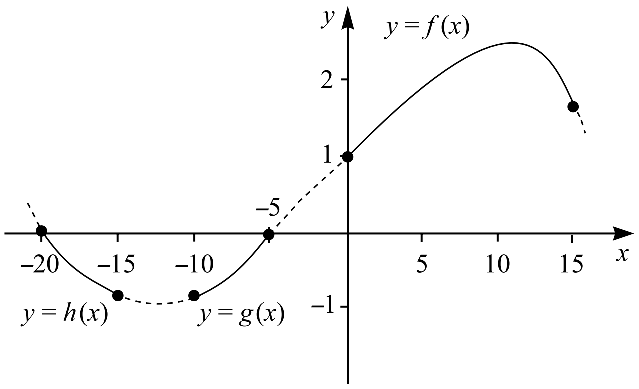
(a) $f(x) = 2 - 3x$

(b) $f(x) = 2x + 2 \quad (x \text{ in } (-1,2))$

(c) $f(x) = 3x^2 + 18x + 15 \quad (x \geq -3)$
(Exercise 2.3(a) on page 7 of this booklet asks for a sketch of the function $f(x) = 3x^2 + 18x + 15$.)

Exercise 4.2

The diagram below gives the graph of a function. The graphs of three functions f, g and h are indicated by the appropriate solid parts of the curve. (The domain of f is $[0, 15]$, of g is $[-10, -5]$ and of h is $[-20, -15]$.)



(a) For each of the functions f, g and h, use the graph to decide whether the function is: increasing, decreasing, neither increasing nor decreasing, one-one, many-one.

(b) Which of the functions f, g and h has an inverse function?

(c) Suppose that the value of $f(x)$ for x in the domain $[0, 15]$ represents the depth in metres of a 15-metre-long swimming pool x metres from one end. Describe in words what happens to the depth of the pool as one goes along it starting from the end where $x = 0$. For what approximate value of x is the pool at its deepest? For what approximate value of x is the pool at its shallowest?

Exercise 4.3

Some of the following 'equations' are not in fact correct. Decide which are correct and which are not. (Try to do this exercise without using your calculator.)

(a) $\tan(\arctan(-2)) = -2$

(b) $\arctan(\tan(-2)) = -2$

(c) $\arcsin(\sin(-\frac{1}{5}\pi)) = \arccos(\cos(-\frac{1}{5}\pi))$

(d) $\arctan(\tan(-\frac{3}{8}\pi)) = \arctan(\tan(\frac{5}{8}\pi))$

(e) $\sin(\arcsin(\pi)) = \pi$

Exercise 4.4

Give the exact value of each of the following expressions, without using your calculator.

(a) $\log_3 1$

(b) $\log_5(\frac{1}{25})$

(c) $\ln(e^{3.1})$

(d) $\log_{10}(\sqrt{1000})$

(e) $e^{\ln 4}$

(f) $2^{3\log_2 5}$

Exercise 4.5

Solve each of the following equations.

(a) $3^x = 70$

(b) $3^x = 81$

Exercise 4.6

Evaluate $\log_3 6$ by first writing $\log_3 6 = x$ and then expressing 6 in terms of x and 3.

Exercise 4.7

Verify each of the following equations.

(a) $\ln(\frac{1}{2}xy^3) = \ln x + 3\ln y - \ln 2$

(b) $\ln\left(\dfrac{3x^2 - 12}{e^{2x}}\right) = \ln 3 + \ln(x-2) + \ln(x+2) - 2x$

Exercise 4.8

A rabbit population is P_n at the start of year n, where

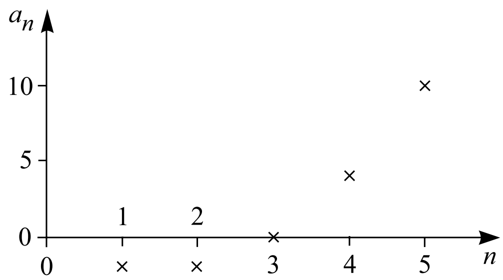
$$P_n = 1400 \times (1.3)^{n-1} + 4200 \quad (n = 1, 2, \ldots).$$

During which year will the population reach more than four times its size at the start of year 1?

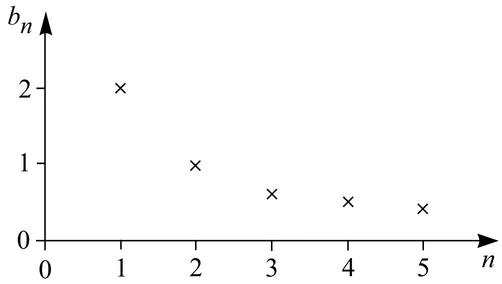
Solutions for Chapter A1

Solution 1.1

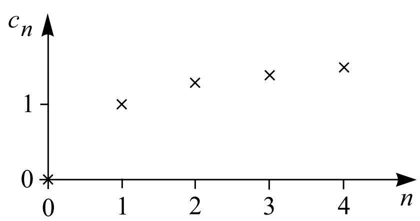
(a) $a_1 = 1^2 - 3 = -2,$
$a_2 = 2^2 - 6 = -2,$
$a_3 = 3^2 - 9 = 0,$
$a_4 = 4^2 - 12 = 4,$
$a_5 = 5^2 - 15 = 10,$
$a_{10} = 10^2 - 30 = 70.$



(b) $b_1 = \frac{2}{1} = 2,$
$b_2 = \frac{2}{2} = 1,$
$b_3 = \frac{2}{3},$
$b_4 = \frac{2}{4} = \frac{1}{2},$
$b_5 = \frac{2}{5},$
$b_{10} = \frac{2}{10} = \frac{1}{5}.$



(c) $c_0 = 0^{1/3} = 0,$
$c_1 = 1^{1/3} = 1,$
$c_2 = 2^{1/3} = 1.260$ (to 3 d.p.),
$c_3 = 3^{1/3} = 1.442$ (to 3 d.p.),
$c_4 = 4^{1/3} = 1.587$ (to 3 d.p.),
$c_9 = 9^{1/3} = 2.080$ (to 3 d.p.).



Solution 1.2

(a) $a_1 = 0,$
$a_2 = a_1^2 + 1 = 0^2 + 1 = 1,$
$a_3 = a_2^2 + 1 = 1^2 + 1 = 2,$
$a_4 = a_3^2 + 1 = 2^2 + 1 = 5,$
$a_5 = a_4^2 + 1 = 5^2 + 1 = 26.$

(b) $b_1 = 1,$
$b_2 = b_1^2 - 1 = 1^2 - 1 = 0,$
$b_3 = b_2^2 - 1 = 0^2 - 1 = -1,$
$b_4 = b_3^2 - 1 = (-1)^2 - 1 = 0,$
$b_5 = b_4^2 - 1 = 0^2 - 1 = -1.$

(c) $c_0 = 0,$

$$c_1 = \frac{1}{1+c_0} = \frac{1}{1+0} = 1,$$

$$c_2 = \frac{1}{1+c_1} = \frac{1}{1+1} = \tfrac{1}{2},$$

$$c_3 = \frac{1}{1+c_2} = \frac{1}{1+\frac{1}{2}} = \tfrac{2}{3},$$

$$c_4 = \frac{1}{1+c_3} = \frac{1}{1+\frac{2}{3}} = \tfrac{3}{5}.$$

Solution 2.1

(a) (i) The first term is $a = 17$ and the common difference is $d = 21 - 17 = 4$. So the recurrence system is

$$x_1 = 17, \quad x_{n+1} = x_n + 4 \quad (n = 1, 2, 3, \ldots).$$

(It would also be correct to take the first term as x_0 rather than x_1, in which case the recurrence system would be

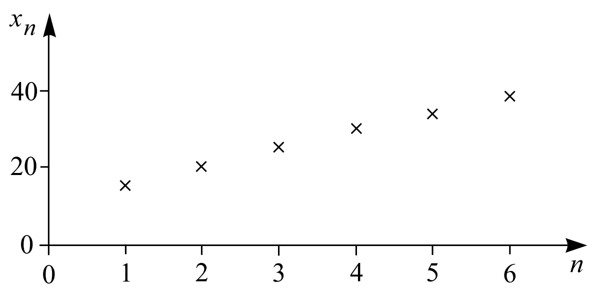
$$x_0 = 17, \quad x_{n+1} = x_n + 4 \quad (n = 0, 1, 2, \ldots).$$

The solution to the next part would then give the values of x_4 and x_5, and the graph would plot the values of x_n for $n = 0, 1, 2, 3, 4, 5$.)

(ii) The next two terms are

$$x_5 = x_4 + 4 = 29 + 4 = 33,$$
$$x_6 = x_5 + 4 = 33 + 4 = 37.$$

The graph is as follows.



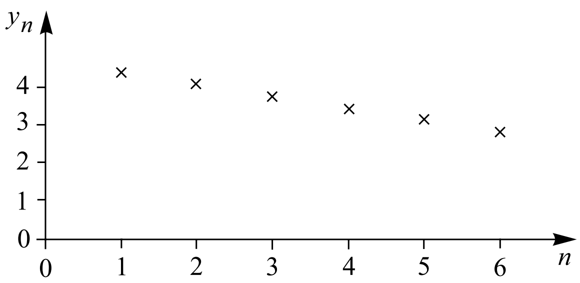
(b) (i) The first term is $a = 4.4$ and the common difference is $d = 4.1 - 4.4 = -0.3$. So the recurrence system is

$$y_1 = 4.4, \quad y_{n+1} = y_n - 0.3 \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$y_6 = y_5 - 0.3 = 3.2 - 0.3 = 2.9,$$
$$y_7 = y_6 - 0.3 = 2.9 - 0.3 = 2.6.$$

The graph is as follows.



Solution 2.2

(a) Since $a = -7$ and $d = 6$, the closed form is

$$\begin{aligned} x_n &= -7 + 6(n-1) \\ &= 6n - 13 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

The 12th term is

$$x_{12} = 6 \times 12 - 13 = 59.$$

(b) Since $a = 9.1$, $d = -0.8$ and the first term is y_0, the closed form is

$$y_n = 9.1 - 0.8n = -0.8n + 9.1 \quad (n = 0, 1, 2, \ldots).$$

The 12th term is

$$y_{11} = 9.1 - 0.8 \times 11 = 0.3.$$

Solution 3.1

(a) (i) The first term is $a = 2$ and the common ratio is $r = 2.8/2 = 1.4$. So the recurrence system is

$$x_1 = 2, \quad x_{n+1} = 1.4x_n \quad (n = 1, 2, 3, \ldots).$$

(It would also be correct to take the first term as x_0 rather than x_1, in which case the recurrence system would be

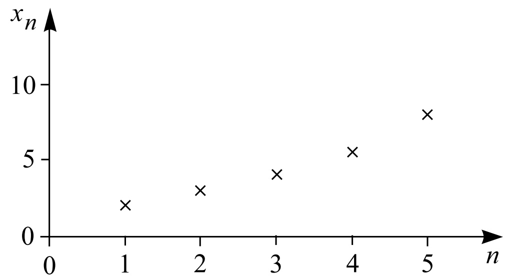
$$x_0 = 2, \quad x_{n+1} = 1.4x_n \quad (n = 0, 1, 2, \ldots).$$

The solution to the next part would then give the values of x_3 and x_4, and the graph would plot the values of x_n for $n = 0, 1, 2, 3, 4$.)

(ii) The next two terms are

$$x_4 = 1.4x_3 = 5.488,$$
$$x_5 = 1.4x_4 = 7.6832 = 7.683 \text{ (to 3 d.p.)}.$$

The graph is as follows.



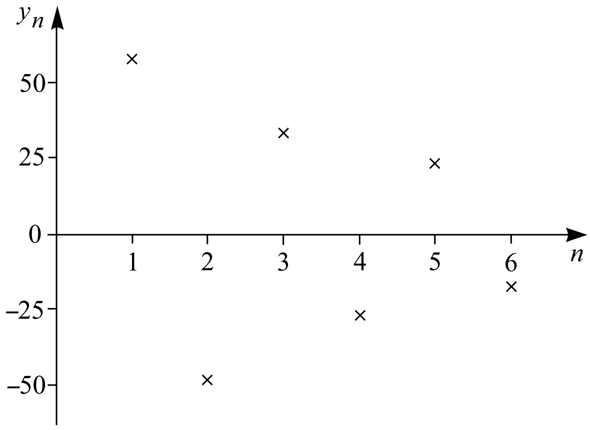
(b) (i) The first term is $a = 64$ and the common ratio is $r = -48/64 = -0.75$. So the recurrence system is

$$y_1 = 64, \quad y_{n+1} = -0.75y_n \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$\begin{aligned} y_5 &= -0.75y_4 = -0.75 \times (-27) = 20.25, \\ y_6 &= -0.75y_5 = -0.75 \times 20.25 = -15.1875 \\ &= -15.188 \text{ (to 3 d.p.)}. \end{aligned}$$

The graph is as follows.



Solution 3.2

(a) Since $a = -3$ and $r = -2.1$, the closed form is

$$x_n = -3(-2.1)^{n-1} \quad (n = 1, 2, 3, \ldots).$$

The 12th term is

$$\begin{aligned} x_{12} &= -3(-2.1)^{11} \\ &= 10\,508.325\,02 = 10\,500 \text{ (to 3 s.f.)}. \end{aligned}$$

(Do not worry if your calculator gives a slightly different answer to ours beyond the eighth or ninth significant figure.)

(b) Since $a = 83$, $r = 0.79$ and the first term is y_0, the closed form is

$$y_n = 83(0.79)^n \quad (n = 0, 1, 2, \ldots).$$

The 12th term is

$$\begin{aligned} y_{11} &= 83(0.79)^{11} \\ &= 6.208\,348\,627 = 6.21 \text{ (to 3 s.f.)}. \end{aligned}$$

Solution 4.1

(a) (i) The first term is $a = 5$. To find r and d, we use the terms $x_1 = 5$, $x_2 = 10$ and $x_3 = 9$. We obtain

$$10 = 5r + d, \quad (1)$$
$$9 = 10r + d. \quad (2)$$

Subtracting equation (1) from equation (2) gives

$$5r = -1;$$

that is, $r = -1/5 = -0.2$, and so, from equation (1),

$$d = 10 - 5r = 10 - 5 \times (-0.2) = 11.$$

So the recurrence system is

$$x_1 = 5, \quad x_{n+1} = -0.2x_n + 11 \quad (n = 1, 2, 3, \ldots).$$

(It would also be correct to take the first term as x_0 rather than x_1, in which case the recurrence system would be

$$x_0 = 5, \quad x_{n+1} = -0.2x_n + 11 \quad (n = 0, 1, 2, \ldots).$$

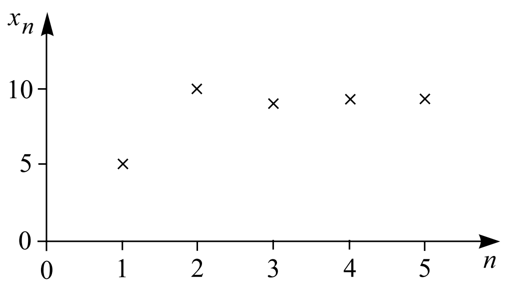
The solution to the next part would then give the values of x_3 and x_4, and the graph would plot the values of x_n for $n = 0, 1, 2, 3, 4$.)

(ii) The next two terms are

$$\begin{aligned} x_4 &= -0.2x_3 + 11 \\ &= -0.2 \times 9 + 11 \\ &= 9.2, \end{aligned}$$

$$\begin{aligned} x_5 &= -0.2x_4 + 11 \\ &= -0.2 \times 9.2 + 11 \\ &= 9.16. \end{aligned}$$

The graph is as follows.



(b) (i) The first term is $a = 3$. To find r and d, we use the terms $y_1 = 3$, $y_2 = 2.9$ and $y_3 = 2.67$. We obtain

$$2.9 = 3r + d, \quad (1)$$
$$2.67 = 2.9r + d. \quad (2)$$

Subtracting equation (2) from equation (1) gives

$$0.1r = 0.23;$$

that is, $r = 0.23/0.1 = 2.3$, and so, from equation (1),

$$d = 2.9 - 3r = 2.9 - 3 \times 2.3 = -4.$$

Hence the recurrence system is

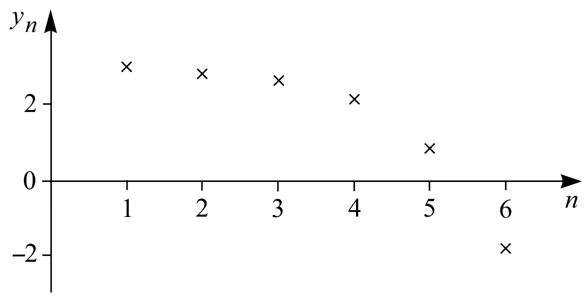
$$y_1 = 3, \quad y_{n+1} = 2.3y_n - 4 \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$\begin{aligned} y_5 &= 2.3y_4 - 4 \\ &= 2.3 \times 2.141 - 4 \\ &= 0.9243 = 0.924 \text{ (to 3 d.p.)}, \end{aligned}$$

$$\begin{aligned} y_6 &= 2.3y_5 - 4 \\ &= 2.3 \times 0.9243 - 4 \\ &= -1.874\,11 = -1.874 \text{ (to 3 d.p.)}. \end{aligned}$$

The graph is as follows.



Solution 4.2

(a) Since $a = 2$, $r = 4$ and $d = 3$, we have

$$\frac{d}{r-1} = \frac{3}{3} = 1,$$

and so the closed form is

$$\begin{aligned} x_n &= (2+1)4^{n-1} - 1 \\ &= 3 \times 4^{n-1} - 1 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

Hence the 9th term is

$$\begin{aligned} x_9 &= 3 \times 4^8 - 1 \\ &= 196\,607 = 196\,600 \text{ (to 4 s.f.)}. \end{aligned}$$

(b) Since $a = 7$, $r = -\frac{1}{3}$ and $d = -1$, we have

$$\frac{d}{r-1} = \frac{-1}{-\frac{1}{3} - 1} = \frac{3}{4}.$$

Since the first term is y_0, the closed form is

$$\begin{aligned} y_n &= \left(7 + \tfrac{3}{4}\right)\left(-\tfrac{1}{3}\right)^n - \tfrac{3}{4} \\ &= \tfrac{31}{4}\left(-\tfrac{1}{3}\right)^n - \tfrac{3}{4} \quad (n = 0, 1, 2, \ldots). \end{aligned}$$

Hence the 9th term is

$$\begin{aligned} y_8 &= \tfrac{31}{4}\left(-\tfrac{1}{3}\right)^8 - \tfrac{3}{4} \\ &= -0.748\ 818\ 778 = -0.7488 \text{ (to 4 s.f.)}. \end{aligned}$$

(c) Since $d = 0$, z_n is a geometric sequence with $a = 2$ and $r = 2.5$, with closed form

$$z_n = 2(2.5)^{n-1} \quad (n = 1, 2, 3, \ldots).$$

Hence the 9th term is

$$\begin{aligned} z_9 &= 2(2.5)^8 \\ &= 3051.757\,813 = 3052 \text{ (to 4 s.f.)}. \end{aligned}$$

(d) Since $\dfrac{-6}{8} = \dfrac{4.5}{-6} = \dfrac{-3.375}{4.5} = -0.75$, the sequence d_n is geometric with $a = 8$ and $r = -0.75$.

(The corresponding recurrence system is

$$d_1 = 8, \quad d_{n+1} = -0.75 d_n \quad (n = 1, 2, 3, \ldots),$$

and the closed form is

$$d_n = 8(-0.75)^{n-1} \quad (n = 1, 2, 3, \ldots).)$$

Solution 4.3

Given that we are dealing with an infinite linear recurrence sequence, a reliable method for this exercise is to compute the parameters a, r and d by solving the pair of simultaneous equations obtained from the first three terms of the sequence. If $r = 1$, then the sequence is arithmetic, and if $d = 0$, then the sequence is geometric. However, if you are asked to decide only whether the sequence is arithmetic, then it might be less work to test whether the differences between any two successive terms are the same. Likewise, just to decide whether the sequence is geometric, it might be less work to test whether the ratios of successive terms are the same.

(a) Since $\dfrac{6.8}{8.5} = \dfrac{5.44}{6.8} = \dfrac{4.352}{5.44} = 0.8$, the sequence a_n is geometric with $a = 8.5$ and $r = 0.8$.

(The corresponding recurrence system is

$$a_1 = 8.5, \quad a_{n+1} = 0.8 a_n \quad (n = 1, 2, 3, \ldots),$$

and the closed form is

$$a_n = 8.5(0.8)^{n-1} \quad (n = 1, 2, 3, \ldots).)$$

(b) Since $6.8 - 8.5 = 5.1 - 6.8 = 3.4 - 5.1 = -1.7$, the sequence b_n is arithmetic with $a = 8.5$ and $d = -1.7$.

(The corresponding recurrence system is

$$b_1 = 8.5, \quad b_{n+1} = b_n - 1.7 \quad (n = 1, 2, 3, \ldots),$$

and the closed form is

$$b_n = -1.7n + 10.2 \quad (n = 1, 2, 3, \ldots).)$$

(c) Here neither the ratio of successive terms nor the difference between them is constant, so the sequence c_n is neither geometric nor arithmetic.

(The corresponding recurrence system is

$$c_1 = 8, \quad c_{n+1} = -1.25 c_n + 4 \quad (n = 1, 2, 3, \ldots),$$

and the closed form is

$$c_n = \tfrac{56}{9}(-1.25)^{n-1} + \tfrac{16}{9} \quad (n = 1, 2, 3, \ldots).)$$

Solution 5.1

(a) As n becomes large, $(-1.09)^n$ is unbounded and alternates in sign (since $-1.09 < -1$), and the same is therefore the case for a_n.

(b) As n becomes large, $0.007n$ becomes arbitrarily large, and so therefore does b_n.

(c) As n becomes large, 0.41^{n-1} becomes arbitrarily small (since $0 < 0.41 < 1$), and so c_n approaches arbitrarily close to 1.99, with all its terms above this value.

Solution 5.2

(a) Since $a = -5$, $r = 0.4$ and $d = 12$, we have

$$\frac{d}{r-1} = \frac{12}{-0.6} = -20,$$

and so the closed form is

$$\begin{aligned} x_n &= \big(-5 + (-20)\big)(0.4)^{n-1} - (-20) \\ &= -25(0.4)^{n-1} + 20 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

As n becomes large, $(0.4)^{n-1}$ becomes arbitrarily small (since $0 < 0.4 < 1$), and so x_n approaches arbitrarily close to 20, with all its terms below this value.

(b) Since $a = 2$, $r = -0.8$ and $d = -27$, we have

$$\frac{d}{r-1} = \frac{-27}{-1.8} = 15,$$

and as the first term is y_0, the closed form is

$$\begin{aligned} y_n &= (2 + 15)(-0.8)^n - 15 \\ &= 17(-0.8)^n - 15 \quad (n = 0, 1, 2, \ldots). \end{aligned}$$

As n becomes large, $(-0.8)^{n-1}$ becomes arbitrarily small and alternates in sign (since $-1 < -0.8 < 0$), and so y_n approaches arbitrarily close to -15, with its terms alternately above and below this value.

Solution 5.3

(a) The sequence is arithmetic (as can be seen by testing the differences of successive terms) with parameters $a = -6$ and $d = 0.2$. So the recurrence system is

$$a_1 = -6, \quad a_{n+1} = a_n + 0.2 \quad (n = 1, 2, 3, \ldots).$$

The closed form is

$$\begin{aligned} a_n &= -6 + 0.2(n-1) \\ &= 0.2n - 6.2 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

As n becomes large, $0.2n$ becomes arbitrarily large, and so therefore does a_n.

(b) The sequence is geometric (as can be seen by testing the ratios of successive terms) with parameters $a = -1200$ and $r = 0.9$. So the recurrence system is

$$b_1 = -1200, \quad b_{n+1} = 0.9b_n \quad (n = 1, 2, 3, \ldots).$$

The closed form is

$$b_n = -1200(0.9)^{n-1} \quad (n = 1, 2, 3, \ldots).$$

As n becomes large, $(0.9)^{n-1}$ becomes arbitrarily small (since $0 < 0.9 < 1$), and so therefore does b_n.

(c) This sequence is neither arithmetic nor geometric, so we use the method of Section 4. The first term is $a = 36$. To find r and d, we use the terms $c_1 = 36$, $c_2 = 18$ and $c_3 = 54$. We obtain

$$18 = 36r + d, \quad (1)$$

$$54 = 18r + d. \quad (2)$$

Subtracting equation (2) from equation (1) gives

$$18r = -36;$$

that is, $r = -36/18 = -2$, and so, from equation (1),

$$d = 18 - 36r = 18 - (-72) = 90.$$

Since $a = 36$, $r = -2$ and $d = 90$, we have

$$\frac{d}{r-1} = \frac{90}{-3} = -30,$$

and so the closed form is

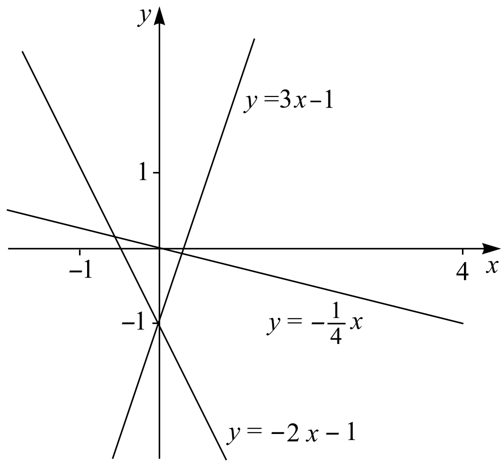
$$\begin{aligned} c_n &= \big(36 + (-30)\big)(-2)^{n-1} - (-30) \\ &= 6(-2)^{n-1} + 30 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

As n becomes large, $(-2)^{n-1}$ is unbounded and alternates in sign, and the same therefore holds for c_n. (Note that the recurrence system, which can be written down once the values of a, r and d are known, is

$$c_1 = 36, \quad c_{n+1} = -2c_n + 90 \quad (n = 1, 2, 3, \ldots).)$$

Solutions for Chapter A2

Solution 1.1



Solution 1.2

(a) The x-intercept is $\frac{3}{2}$ (the value of x when $y = 0$). The y-intercept is 2 (the value of y when $x = 0$).

(b) The x-intercept and y-intercept are both 0.

Solution 1.3

(a) The equation is

$$y - 2 = 4\big(x - (-3)\big),$$

which can be rearranged as $y = 4x + 14$.

(b) From $(-3, 2)$ to $(1, -4)$, the rise is $-4 - 2 = -6$ and the run is $1 - (-3) = 4$, so the slope of the line is $-6/4 = -\frac{3}{2}$. Since the line passes through the point $(-3, 2)$, its equation can be written as

$$y - 2 = -\tfrac{3}{2}(x - (-3)),$$

which can be rearranged as $y = -\frac{3}{2}x - \frac{5}{2}$.

(c) From $(-3, 2)$ to $(7, 2)$, the rise is $2 - 2 = 0$ and the run is $7 - (-3) = 10$, so the slope of the line is $0/10 = 0$. (In general, if two points have the same y-coordinate, then the line joining them has slope 0; that is, it is horizontal.) Since the line passes through the point $(-3, 2)$, its equation can be written as

$$y - 2 = 0(x - (-3)),$$

which can be rearranged as $y = 2$.

(You might have obtained this answer just by noticing that the y-coordinates of the two points are both 2.)

(d) The line passes through the points $(-3, 0)$ and $(0, 3)$. So its rise is $3 - 0 = 3$ and its run is $0 - (-3) = 3$, and its slope is thus $3/3 = 1$.

Since the line passes through $(-3, 0)$, its equation can be written as

$$y - 0 = 1(x - (-3)),$$

which can be rearranged as $y = x + 3$.

Solution 1.4

(a) From $A(-1, -2)$ to $B(3, 5)$, the rise is $5 - (-2) = 7$ and the run is $3 - (-1) = 4$. Hence the slope of AB is $\frac{7}{4}$.

(b) Each line which is perpendicular to AB has slope $-1/(\frac{7}{4}) = -\frac{4}{7}$.

(c) Since the required line passes through the point $(-1, -2)$ and has slope $-\frac{4}{7}$, its equation can be written as

$$y - (-2) = -\tfrac{4}{7}(x - (-1)),$$

which can be rearranged as $y = -\frac{4}{7}x - \frac{18}{7}$.

Solution 1.5

The slope of the line $y = -\frac{3}{5}x + \frac{2}{5}$ is $-\frac{3}{5}$, so any perpendicular line has slope $-1/(-\frac{3}{5}) = \frac{5}{3}$. As the required line passes through the point $(-1, 1)$ and has slope $\frac{5}{3}$, its equation can be written as

$$y - 1 = \tfrac{5}{3}(x - (-1)),$$

which can be rearranged as $y = \frac{5}{3}x + \frac{8}{3}$.

Solution 1.6

(a) The two lines $y = -2x + 1$ and $y = 7x - 2$ meet when

$$-2x + 1 = 7x - 2; \quad \text{that is,} \quad x = \tfrac{1}{3}.$$

The corresponding value of y is

$$y = -2 \times \tfrac{1}{3} + 1 = \tfrac{1}{3},$$

so the lines intersect at $(\frac{1}{3}, \frac{1}{3})$.

(b) The vertical line $x = 3$ meets the line $y = 7x - 2$ at the point on the latter which has x-coordinate 3. The y-coordinate of this point is

$$y = 7 \times 3 - 2 = 19,$$

so the lines intersect at $(3, 19)$.

Solution 2.1

(a) The square of the distance between the points $(8, 9)$ and $(2, 7)$ is

$$(8 - 2)^2 + (9 - 7)^2 = 6^2 + 2^2 = 40.$$

Hence the distance is $\sqrt{40} = 2\sqrt{10} \simeq 6.32$.

(b) The square of the distance between the points $(-5, 0)$ and $(-2, -3)$ is

$$(-5 - (-2))^2 + (0 - (-3))^2 = (-3)^2 + 3^2 = 18.$$

Hence the distance is $\sqrt{18} = 3\sqrt{2} \simeq 4.24$.

Solution 2.2

(a) The centre is at $(0, 0)$ and the radius is 6, so the equation is $x^2 + y^2 = 36$.

(b) The centre is at $(-2, 0)$ and the radius is $\frac{1}{3}$, so the equation is $(x - (-2))^2 + y^2 = \frac{1}{9}$; that is, $(x + 2)^2 + y^2 = \frac{1}{9}$.

(c) The centre is at $(4, -1)$ and the radius is $\sqrt{3}$, so the equation is $(x - 4)^2 + (y - (-1))^2 = 3$; that is, $(x - 4)^2 + (y + 1)^2 = 3$.

Solution 2.3

(a) $(x - 4)^2 + (y - 2)^2 = 9$: centre is $(4, 2)$, radius is 3.

(b) $(x + 5)^2 + (y - 3)^2 = 64$: centre is $(-5, 3)$, radius is 8.

(c) $(x - \sqrt{2})^2 + y^2 = 8$: centre is $(\sqrt{2}, 0)$, radius is $\sqrt{8} = 2\sqrt{2} \simeq 2.83$.

Solution 2.4

(a) The line segment AB has slope $(-7 - 3) \div (1 - (-1)) = -5$. Its midpoint M has coordinates $(\frac{1}{2}(-1 + 1), \frac{1}{2}(3 - 7)) = (0, -2)$.

(b) The line through M perpendicular to AB has slope $-1/(-5) = \frac{1}{5}$ (using the perpendicularity condition from Subsection 1.1 of Chapter A2). Since it passes through $M(0, -2)$, its equation is

$$y - (-2) = \tfrac{1}{5}(x - 0); \quad \text{that is,} \quad y = \tfrac{1}{5}x - 2.$$

(c) The line segment BC has slope $(13 - (-7)) \div (5 - 1) = 5$ and midpoint $(\frac{1}{2}(1 + 5), \frac{1}{2}(-7 + 13)) = (3, 3)$. Its perpendicular bisector therefore has slope $-\frac{1}{5}$ and passes through $(3, 3)$. Its equation is

$$y - 3 = -\tfrac{1}{5}(x - 3); \quad \text{that is,} \quad y = -\tfrac{1}{5}x + \tfrac{18}{5}.$$

(d) The two perpendicular bisectors, $y = \frac{1}{5}x - 2$ and $y = -\frac{1}{5}x + \frac{18}{5}$, intersect when

$$\tfrac{1}{5}x - 2 = -\tfrac{1}{5}x + \tfrac{18}{5}; \quad \text{that is,} \quad x = 14.$$

The corresponding value of y is $y = \frac{1}{5} \times 14 - 2 = \frac{4}{5}$, so D is the point $(14, \frac{4}{5})$.

(e) The radius r of the circle through the points A, B and C is the distance between D and B (or A, or C), so it is given by

$$\begin{aligned}
r^2 &= (14 - 1)^2 + (\tfrac{4}{5} - (-7))^2 \\
&= 13^2 + \left(\tfrac{39}{5}\right)^2 \\
&= 13^2 \left(1 + (\tfrac{3}{5})^2\right) \\
&= 13^2 \times \tfrac{34}{25} \\
&= \frac{5746}{25}.
\end{aligned}$$

The radius is $r = \frac{1}{5}\sqrt{5746} = \frac{13}{5}\sqrt{34} \simeq 15.16$. The equation of the circle is

$$(x-14)^2 + (y - \tfrac{4}{5})^2 = \frac{5746}{25}.$$

Solution 2.5

The line segment AB has slope $(4-0) \div (5-(-3)) = \frac{1}{2}$ and midpoint $(\frac{1}{2}(-3+5), \frac{1}{2}(0+4)) = (1,2)$. Its perpendicular bisector therefore has slope -2 and passes through $(1,2)$. Its equation is

$$y - 2 = -2(x-1); \quad \text{that is,} \quad y = -2x + 4.$$

Similarly, the line segment BC has slope $(10-4) \div (2-5) = -2$ and midpoint $(\frac{1}{2}(5+2), \frac{1}{2}(4+10)) = (\frac{7}{2}, 7)$. Its perpendicular bisector therefore has slope $\frac{1}{2}$ and passes through $(\frac{7}{2}, 7)$. Its equation is

$$y - 7 = \tfrac{1}{2}(x - \tfrac{7}{2}); \quad \text{that is,} \quad y = \tfrac{1}{2}x + \tfrac{21}{4}.$$

The two perpendicular bisectors, $y = -2x + 4$ and $y = \frac{1}{2}x + \frac{21}{4}$, intersect when

$$-2x + 4 = \tfrac{1}{2}x + \tfrac{21}{4}; \quad \text{that is,} \quad x = -\tfrac{1}{2}.$$

The corresponding value of y is $y = -2(-\frac{1}{2}) + 4 = 5$, so the centre D of the circle is at $(-\frac{1}{2}, 5)$.

The radius r is the distance between D and B (or A, or C), so it is given by

$$r^2 = (-\tfrac{1}{2} - 5)^2 + (5-4)^2 = \tfrac{125}{4}.$$

The radius is $r = \frac{1}{2}\sqrt{125} = \frac{5}{2}\sqrt{5} \simeq 5.59$. The equation of the circle is

$$(x + \tfrac{1}{2})^2 + (y-5)^2 = \tfrac{125}{4}.$$

Solution 2.6

(a) Comparing $x^2 + 20x$ with the identity

$$x^2 + 2px = (x+p)^2 - p^2$$

from Subsection 2.3 of Chapter A2, we can match the left-hand side by putting $2p = 20$; that is, $p = 10$. Putting this value also into the right-hand side, we obtain the completed-square form

$$x^2 + 20x = (x+10)^2 - 10^2 = (x+10)^2 - 100.$$

(b) Comparing $x^2 - 11x$ with the identity

$$x^2 + 2px = (x+p)^2 - p^2,$$

we can match the left-hand side by putting $2p = -11$; that is, $p = -\frac{11}{2}$. Putting this value also into the right-hand side, we obtain the completed-square form

$$\begin{aligned} x^2 - 11x &= (x - \tfrac{11}{2})^2 - (-\tfrac{11}{2})^2 \\ &= (x - \tfrac{11}{2})^2 - \tfrac{121}{4}. \end{aligned}$$

(c) Comparing $y^2 + 7y$ with the identity

$$y^2 + 2py = (y+p)^2 - p^2,$$

we can match the left-hand side by putting $2p = 7$; that is, $p = \frac{7}{2}$. Putting this value also into the right-hand side, we obtain the completed-square form

$$y^2 + 7y = (y + \tfrac{7}{2})^2 - (\tfrac{7}{2})^2 = (y + \tfrac{7}{2})^2 - \tfrac{49}{4}.$$

Solution 2.7

(a) The method involves completing the square for the x terms and then for the y terms.

Comparing $x^2 + 12x$ with the identity

$$x^2 + 2px = (x+p)^2 - p^2$$

from Subsection 2.3, we can match the left-hand side by putting $2p = 12$; that is, $p = 6$. Putting this value also into the right-hand side, we obtain the completed-square form

$$x^2 + 12x = (x+6)^2 - 6^2 = (x+6)^2 - 36.$$

Comparing $y^2 - 6y$ with the identity

$$y^2 + 2py = (y+p)^2 - p^2,$$

we can match the left-hand side by putting $2p = -6$; that is, $p = -3$. Putting this value also into the right-hand side, we obtain the completed-square form

$$\begin{aligned} y^2 - 6y &= (y-3)^2 - (-3)^2 \\ &= (y-3)^2 - 9. \end{aligned}$$

In terms of these completed-square forms, the equation

$$x^2 + 12x + y^2 - 6y + 40 = 0$$

becomes

$$(x+6)^2 - 36 + (y-3)^2 - 9 + 40 = 0.$$

On collecting the number terms and rearranging, we obtain

$$(x+6)^2 + (y-3)^2 = 5 = (\sqrt{5})^2.$$

Hence the equation is indeed that of a circle, which has centre $(-6, 3)$ and radius $\sqrt{5}$.

(b) Comparing $x^2 - 5x$ with the identity

$$x^2 + 2px = (x+p)^2 - p^2,$$

we can match the left-hand side by putting $2p = -5$; that is, $p = -\frac{5}{2}$. Putting this value also into the right-hand side, we obtain the completed-square form

$$\begin{aligned} x^2 - 5x &= (x - \tfrac{5}{2})^2 - (-\tfrac{5}{2})^2 \\ &= (x - \tfrac{5}{2})^2 - \tfrac{25}{4}. \end{aligned}$$

Comparing $y^2 + 3y$ with the identity

$$y^2 + 2py = (y+p)^2 - p^2,$$

we can match the left-hand side by putting $2p = 3$; that is, $p = \frac{3}{2}$. Putting this value also into the right-hand side, we obtain the completed-square form

$$\begin{aligned} y^2 + 3y &= (y+\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 \\ &= (y+\tfrac{3}{2})^2 - \tfrac{9}{4}. \end{aligned}$$

In terms of these completed-square forms, the equation

$$x^2 - 5x + y^2 + 3y - \tfrac{1}{2} = 0$$

becomes

$$(x-\tfrac{5}{2})^2 - \tfrac{25}{4} + (y+\tfrac{3}{2})^2 - \tfrac{9}{4} - \tfrac{1}{2} = 0.$$

On collecting the number terms and rearranging, we obtain

$$(x-\tfrac{5}{2})^2 + (y+\tfrac{3}{2})^2 = 9 = 3^2.$$

Hence the equation is indeed that of a circle, which has centre at $(\frac{5}{2}, -\frac{3}{2})$ and radius 3.

Solution 2.8

(a) Using the equation of the line, $y = -2x - 1$, to substitute for y in the equation of the circle, $(x+3)^2 + (y-4)^2 = 29$, we have

$$(x+3)^2 + (-2x-1-4)^2 = 29.$$

On multiplying out the brackets and simplifying, we obtain the successive equations

$$\begin{aligned} &(x+3)^2 + (-2x-5)^2 = 29, \\ &x^2 + 6x + 9 + 4x^2 + 20x + 25 = 29, \\ &5x^2 + 26x + 5 = 0. \end{aligned}$$

The last equation factorises as $(5x+1)(x+5) = 0$, from which the solutions are $x = -\frac{1}{5}$ and $x = -5$. Using each of these values in turn in the equation of the line, we obtain respectively $y = -2(-\frac{1}{5}) - 1 = -\frac{3}{5}$ and $y = -2(-5) - 1 = 9$. The line cuts the circle at the two points $(-\frac{1}{5}, -\frac{3}{5})$ and $(-5, 9)$.

(b) Using the equation of the line, $y = 2x - 3$, to substitute for y in the equation of the circle, $(x+3)^2 + (y-4)^2 = 29$, we have

$$(x+3)^2 + (2x-3-4)^2 = 29.$$

On multiplying out the brackets and simplifying, we obtain the successive equations

$$\begin{aligned} &(x+3)^2 + (2x-7)^2 = 29, \\ &x^2 + 6x + 9 + 4x^2 - 28x + 49 = 29, \\ &5x^2 - 22x + 29 = 0. \end{aligned}$$

This equation is of the form $ax^2 + bx + c = 0$ with $b^2 - 4ac = (-22)^2 - 4 \times 5 \times 29 = -96$.

Since this is negative, the quadratic equation formula shows that there are no real roots. Therefore there are no points which lie on both the line and circle.

Solution 3.1

An angle of 360° is the same as one of 2π radians.

(a) (i) 4π (ii) $\frac{3}{4}\pi$ (iii) $\frac{7}{6}\pi$

(b) (i) 540° (ii) 240° (iii) 315°

Solution 3.2

(a) (i) $\sin(720°) = 0$, $\cos(720°) = 1$, $\tan(720°) = 0$.

(ii) $\sin(135°) = \frac{1}{2}\sqrt{2}$, $\cos(135°) = -\frac{1}{2}\sqrt{2}$, $\tan(135°) = -1$.

(iii) $\sin(210°) = -\frac{1}{2}$, $\cos(210°) = -\frac{1}{2}\sqrt{3}$, $\tan(210°) = \frac{1}{3}\sqrt{3}$.

(b) (i) $\sin(3\pi) = 0$, $\cos(3\pi) = -1$, $\tan(3\pi) = 0$.

(ii) $\sin(\frac{4}{3}\pi) = -\frac{1}{2}\sqrt{3}$, $\cos(\frac{4}{3}\pi) = -\frac{1}{2}$, $\tan(\frac{4}{3}\pi) = \sqrt{3}$.

(iii) $\sin(\frac{7}{4}\pi) = -\frac{1}{2}\sqrt{2}$, $\cos(\frac{7}{4}\pi) = \frac{1}{2}\sqrt{2}$, $\tan(\frac{7}{4}\pi) = -1$.

Solution 3.3

(a) Since $BC = 9$ and $\angle A = 29°$, we have $\angle B = 90° - 29° = 61°$ and

$$AC = 9\tan 61° \simeq 16.2,$$

$$AB = \frac{9}{\sin 29°} \simeq 18.6.$$

(b) By Pythagoras' Theorem, we have $AB^2 = 16^2 + 12^2 = 400$, so

$$AB = \sqrt{400} = 20.$$

The sine of $\angle B$ is $\frac{12}{20} = \frac{3}{5}$. Applying the 'inverse sine' facility on a calculator to this gives the result $\angle B \simeq 36.9°$. Hence the remaining angle is given by $\angle A = 90° - \angle B \simeq 53.1°$.

(c) Using the formula for the distance between two points, we have

$$AB^2 = (\tfrac{1}{3} - 9)^2 + (-2-2)^2 = \frac{820}{9},$$

so $AB = \sqrt{\dfrac{820}{9}} \simeq 9.55$. Similarly, we have

$$BC^2 = (9-0)^2 + (2-1)^2 = 82,$$

so $BC = \sqrt{82} \simeq 9.06$, and

$$AC^2 = (\tfrac{1}{3} - 0)^2 + (-2-1)^2 = \frac{82}{9},$$

so $AC = \sqrt{\dfrac{82}{9}} \simeq 3.02$.

The cosine of $\angle A$ is

$$\sqrt{\frac{82}{9}} \div \sqrt{\frac{820}{9}} = \sqrt{\frac{82}{820}}$$
$$= \sqrt{\frac{1}{10}} \simeq 0.316.$$

Applying the 'inverse cosine' facility on a calculator to this gives the result $\angle A \simeq 71.6°$. Hence the remaining angle is given by $\angle B = 90° - \angle A \simeq 18.4°$.

Solution 3.4

The slope of the line is $\tan 60°$ (as is remarked in Section 3 of Chapter A2), and $\tan 60° = \sqrt{3}$. Since the line passes through the point $(4, 5)$, its equation can be written as

$$y - 5 = \sqrt{3}(x - 4),$$

which can be rearranged as $y = \sqrt{3}x + 5 - 4\sqrt{3}$.

Solution 4.1

(a) The line has slope $m = 4$ and passes through $(x_1, y_1) = (-3, 2)$. Hence the parametric equation expressions

$$x = t + x_1, \quad y = mt + y_1$$

lead to

$$x = t - 3, \quad y = 4t + 2.$$

(In this and the remaining parts there are many correct alternative parametric equations.)

(b) The line passes through $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (2, -2)$. Hence the parametric equation expressions

$$x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1)$$

lead to

$$x = 3t - 1, \quad y = -5t + 3.$$

(Note that if the points had been taken in the other order, then the parametric equations would have been

$$x = -3t + 2, \quad y = 5t - 2.)$$

(c) The line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ can be parametrised as

$$x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1),$$

with t in the range $0 \le t \le 1$. Here $(x_1, y_1) = (-6, 5)$ and $(x_2, y_2) = (-2, -1)$, so the parametrisation is

$$x = 4t - 6, \quad y = -6t + 5 \quad (0 \le t \le 1).$$

(Note that if the points had been taken in the other order, then the parametrisation would have been

$$x = -4t - 2, \quad y = 6t - 1 \quad (0 \le t \le 1).)$$

Solution 4.2

If $x = 2t - 7$, then $t = \frac{1}{2}(x + 7)$. Hence we have

$$y = -6t + 3 = -6 \times \tfrac{1}{2}(x + 7) + 3 = -3x - 18;$$

that is, the equation of the line is $y = -3x - 18$.

Solution 4.3

(a) The circle can be described by

$$x = -1 + 4\cos t, \quad y = 5 + 4\sin t \quad (0 \le t \le 2\pi).$$

(In this and the next part there are many correct alternative parametric equations.)

(b) The semi-circle has centre $(a, b) = (1, -1)$ and radius $r = 2$. The whole circle has parametric equations

$$x = 1 + 2\cos t, \quad y = -1 + 2\sin t (0 \le t \le 2\pi).$$

In order to describe the semi-circle shown, an appropriate range for t is $\frac{1}{2}\pi \le t \le \frac{3}{2}\pi$, since the diameter is perpendicular to the x-axis. (Another suitable range is $-\frac{3}{2}\pi \le t \le -\frac{1}{2}\pi$.)

Solutions for Chapter A3

Solution 1.1

(a) $f(x) = x^3$ $(x < 0)$ and $f(-1) = -1$.

(b) $f(x) = 2 - x^2$ $(-2 \le x \le 0)$ and $f(-1) = 1$.

Solution 1.2

(a) $[-3, -2)$ is half-open (or half-closed).

(b) $[-3, \infty)$ is closed.

(c) $(-3, 2)$ is open.

(d) $(-\infty, 2)$ is open.

Solution 1.3

-2 is included in the intervals $(-5, 0)$ and $(-3, -2]$.

Solution 1.4

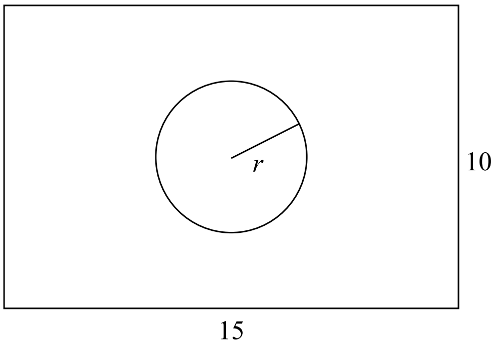
(a) $[0, \infty)$

(b) The expression $\sqrt{x - 1}$ is defined for $x - 1 \ge 0$, which is equivalent to $x \ge 1$. Thus $(\sqrt{x - 1})^3$ is defined for $x \ge 1$. Thus, by the domain convention, the domain is $[1, \infty)$.

(c) The expression $x/(x + 1)$ gives a real number for all x in $\mathbb{R}$ except $x = -1$. The expression $\sqrt{x + 2}$ is defined for $x + 2 \ge 0$, which is equivalent to $x \ge -2$. The expression $3/\sqrt{x + 2}$ is not defined when $x = -2$, as one cannot divide by 0, so it is only defined for $x > -2$. Thus the domain of the function with rule $f(x) = x/(x + 1) + 3/\sqrt{x + 2}$ is $(-2, \infty)$ excluding -1. (This consists of the two open intervals $(-2, -1)$ and $(-1, \infty)$.)

Solution 2.1

(a) As the diameter of the pond (which equals $2r$) cannot exceed the smaller of the dimensions of the plot, namely 10 metres, and r cannot be negative, the largest closed interval of possible values for r is $[0, 5]$.



(b) The area of the lawn is the area of the rectangular plot less the area of the pond, so

$$\begin{aligned} A &= 10 \times 15 - \pi r^2 \\ &= 150 - \pi r^2. \end{aligned}$$

(c) Two-thirds of the area of the plot is 100m^2, so r must satisfy

$$150 - \pi r^2 = 100;$$

that is,

$$r^2 = \frac{50}{\pi}.$$

The solutions of this equation are

$$r = \pm\sqrt{\frac{50}{\pi}};$$

that is, $r \simeq \pm 3.99$. Of these solutions, only $r \simeq 3.99$ lies in the interval $[0, 5]$. Thus the solution of the given problem is 3.99 metres.

Solution 2.2

(a) Using the identity $x^2 + 2px = (x + p)^2 - p^2$ from Chapter A2 with $p = -7$, we obtain:

$$\begin{aligned} x^2 - 14x + 30 &= (x - 7)^2 - 49 + 30 \\ &= (x - 7)^2 - 19. \end{aligned}$$

(b) First take out the factor 3, to give

$$3x^2 + 5x + 1 = 3(x^2 + \tfrac{5}{3}x + \tfrac{1}{3}),$$

and then use the identity $x^2 + 2px = (x + p)^2 - p^2$ with $p = \frac{5}{6}$:

$$\begin{aligned} 3x^2 + 5x + 1 &= 3((x + \tfrac{5}{6})^2 - \tfrac{25}{36} + \tfrac{1}{3}) \\ &= 3((x + \tfrac{5}{6})^2 - \tfrac{13}{36}) \\ &= 3(x + \tfrac{5}{6})^2 - \tfrac{13}{12}. \end{aligned}$$

(c) First take out the factor -4, to give

$$-4x^2 + x + 2 = -4(x^2 - \tfrac{1}{4}x - \tfrac{1}{2}),$$

and then use the identity $x^2 + 2px = (x + p)^2 - p^2$ with $p = -\frac{1}{8}$:

$$\begin{aligned} -4x^2 + x + 2 &= -4((x - \tfrac{1}{8})^2 - \tfrac{1}{64} - \tfrac{1}{2}) \\ &= -4((x - \tfrac{1}{8})^2 - \tfrac{33}{64}) \\ &= -4(x - \tfrac{1}{8})^2 + \tfrac{33}{16}. \end{aligned}$$

Solution 2.3

(a) First, we express the rule of f in completed-square form:

$$\begin{aligned} f(x) &= 3x^2 + 18x + 15 \\ &= 3\,(x^2 + 6x + 5) \\ &= 3\,((x + 3)^2 - 9 + 5) \\ &= 3\,((x + 3)^2 - 4) \\ &= 3\,(x + 3)^2 - 12. \end{aligned}$$

Therefore, the graph of f can be obtained from the graph of $y = x^2$ by performing:

- ◇ a y-scaling with factor 3;
- ◇ a horizontal translation by 3 units to the left;
- ◇ a vertical translation by 12 units downwards.

(The order of performing the y-scaling and the horizontal translation can be reversed.)

The y-intercept is

$$f(0) = 15,$$

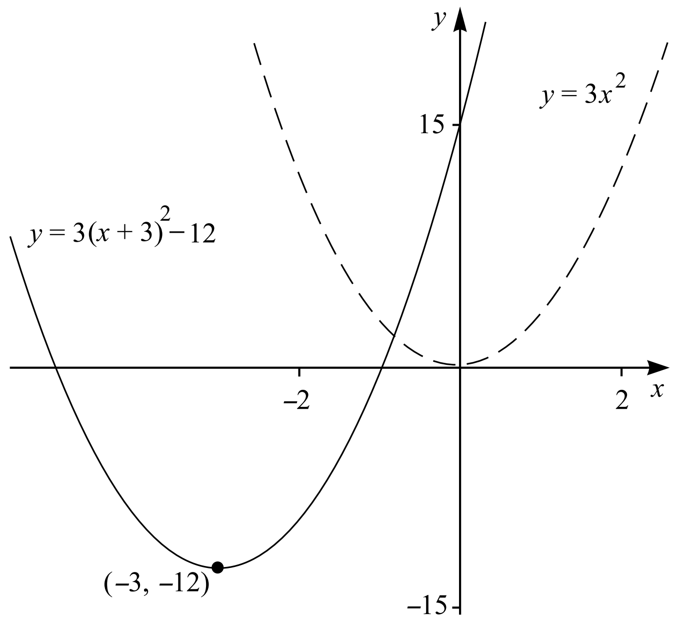
and the x-intercepts are $x = -5$ and $x = -1$. These x-intercepts are found by solving the equation

$$f(x) = 3x^2 + 18x + 15 = 0,$$

which is equivalent to

$$x^2 + 6x + 5 = (x + 5)(x + 1) = 0.$$

The graph of f is the unbroken curve in the following diagram.



(b) First we express the rule of f in completed-square form:

$$\begin{aligned} f(x) &= -3x^2 + 2x - \tfrac{7}{3} \\ &= -3\left(x^2 - \tfrac{2}{3}x + \tfrac{7}{9}\right) \\ &= -3\left((x - \tfrac{1}{3})^2 - \tfrac{1}{9} + \tfrac{7}{9}\right) \\ &= -3\left((x - \tfrac{1}{3})^2 + \tfrac{2}{3}\right) \\ &= -3\left(x - \tfrac{1}{3}\right)^2 - 2. \end{aligned}$$

Therefore, the graph of f can be obtained from the graph of $y = x^2$ by performing:

- ◇ a y-scaling with factor -3;
- ◇ a horizontal translation by $\frac{1}{3}$ of a unit to the right;
- ◇ a vertical translation by 2 units downwards.

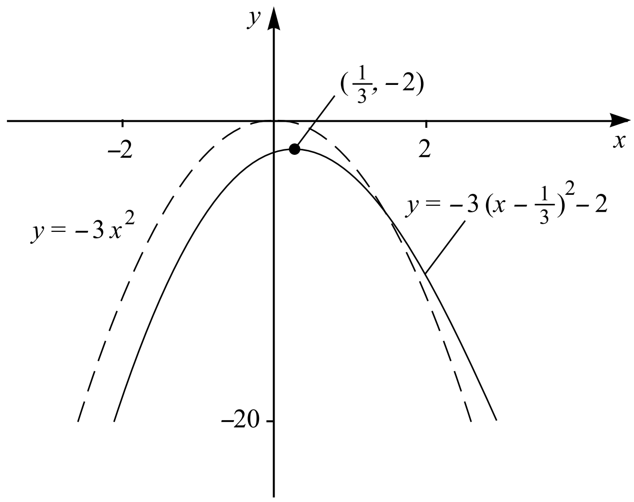
(The order of performing the y-scaling and the horizontal translation can be reversed.)

The y-intercept is

$$f(0) = -\tfrac{7}{3},$$

and the curve never meets the x-axis because $f(x) \leq -2$ for all x.

The graph of f is the unbroken curve in the following diagram.



Solution 2.4

(a) The graph of the function $f(x) = 3x^2 + 18x + 15$ was sketched in Solution 2.3(a) above. The graph of the function

$$f(x) = 3x^2 + 18x + 15 \quad (-2 \leq x \leq 0)$$

consists of that part of the sketch for the values of x in the interval $[-2, 0]$. Thus the image set is $[f(-2), f(0)] = [-9, 15]$.

(b) The graph of the function

$$f(x) = 3x^2 + 18x + 15 \quad (-4 \leq x \leq -1)$$

consists of that part of the sketch for the values of x in the interval $[-4, -1]$. Thus the image set is $[f(-3), f(-1)] = [-12, 0]$.

Solution 2.5

(a) Solving the equation

$$|3x + 7| - 4 = 0$$

is equivalent to solving

$$|3x + 7| = 4.$$

This is equivalent to the two equations

$$3x + 7 = 4 \quad \text{and} \quad 3x + 7 = -4,$$

giving the solutions $x = -1$ and $x = -\frac{11}{3}$.

(b) Solving the equation

$$|3x - 7| + 4 = 0$$

is equivalent to solving

$$|3x - 7| = -4.$$

However, $|3x - 7|$ can never be negative, so the equation has no solutions.

Solution 2.6

The graph of

$$f(x) = \tfrac{1}{2}|x+2| - \tfrac{3}{2}$$

can be obtained from the graph of $y = |x|$ by performing:

- ◇ a y-scaling with factor $\frac{1}{2}$;
- ◇ a horizontal translation by 2 units to the left;
- ◇ a vertical translation by $\frac{3}{2}$ units downwards.

(The order of performing the y-scaling and the horizontal translation can be reversed.)

The y-intercept is

$$f(0) = \tfrac{1}{2}|0+2| - \tfrac{3}{2} = 1 - \tfrac{3}{2} = -\tfrac{1}{2},$$

and the x-intercepts are $x = -5$ and $x = 1$. These x-intercepts are found by solving the equation

$$f(x) = \tfrac{1}{2}|x+2| - \tfrac{3}{2} = 0;$$

that is,

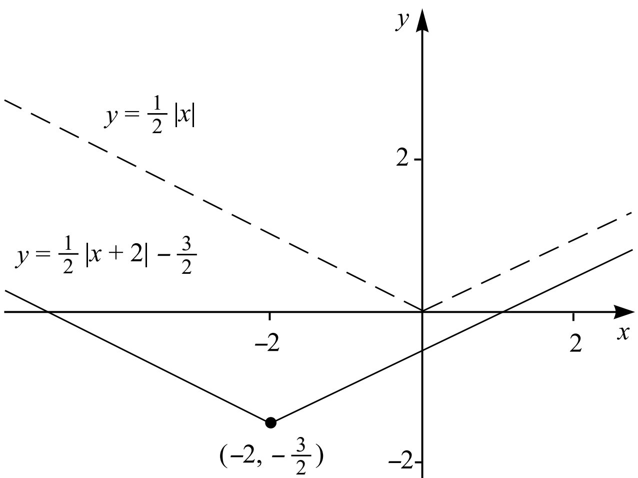
$$\tfrac{1}{2}|x+2| = \tfrac{3}{2},$$

which is equivalent to the two equations

$$\tfrac{1}{2}(x+2) = \tfrac{3}{2} \quad \text{and} \quad \tfrac{1}{2}(x+2) = -\tfrac{3}{2}.$$

The solutions to these are $x = 1$ and $x = -5$.

The graph of f is the unbroken curve in the following diagram.

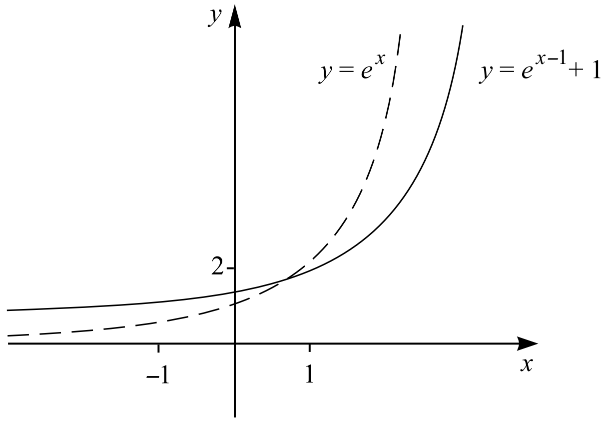


Solution 3.1

The graph of the function $g(x) = e^{x-1} + 1$ can be obtained from the graph of $f(x) = e^x$ by performing:

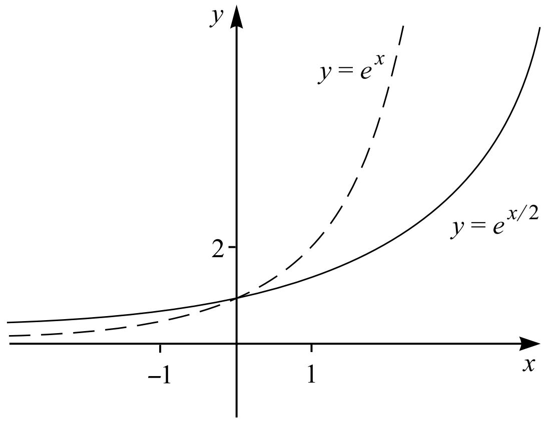
- ◇ a horizontal translation by 1 unit to the right;
- ◇ a vertical translation by 1 unit upwards.

(Note that the graph of g can also be obtained by applying a y-scaling by a factor $e^{-1} = 1/e$, followed by a vertical translation by 1 unit upwards, since $e^{x-1} + 1 = e^{-1} \times e^x + 1$.) The graph is as follows.



Solution 3.2

(a) The graph of $y = e^{x/2}$ can be obtained from the graph of $y = e^x$ by performing an x-scaling with factor 2. This gives the following graph.



(b) The graph of $y = \sin(\pi x)$ can be obtained from the graph of $y = \sin x$ by performing an x-scaling with factor $1/\pi$. This gives the following graph. Note that since $\sin x = 0$ when

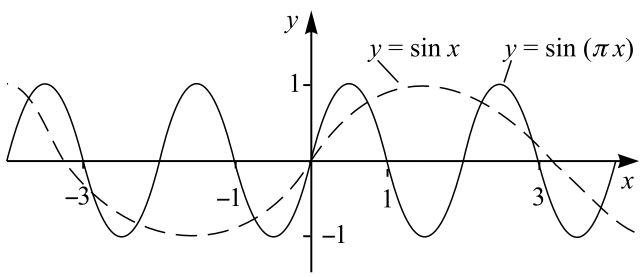
$$x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots,$$

we have $\sin(\pi x) = 0$ when

$$\pi x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots,$$

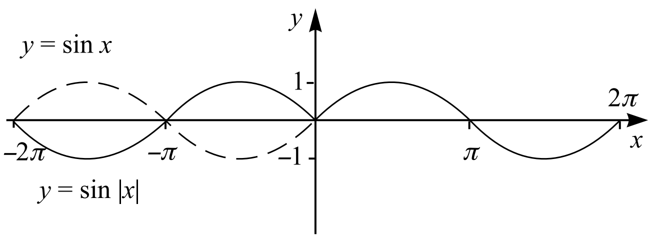
which, on dividing each side by π, gives the x-intercepts

$$x = 0, \pm 1, \pm 2, \pm 3, \ldots.$$



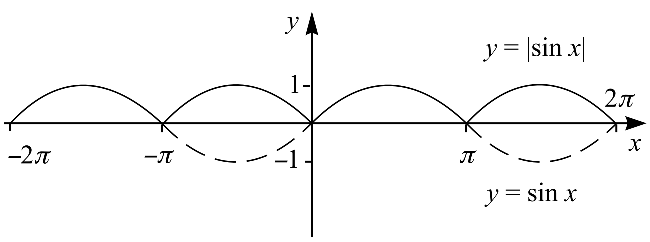
Solution 3.3

(a) For positive values of x, $|x|$ equals x, so the graph of $y = \sin|x|$ is the same as the graph of $y = \sin x$ for these values. For negative values of x, $|x|$ equals $-x$, so the graph of $y = \sin|x|$ for these values is the reflection in the y-axis of the graph of $y = \sin x$ for positive x. This gives the following graph.



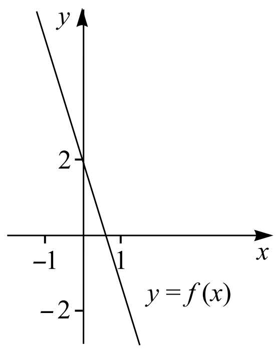
(An alternative way of dealing with negative values of x is to note that for such x we have $\sin|x| = \sin(-x) = -\sin x$, so the graph of $y = \sin|x|$ for these values can also be obtained by reflecting the graph of $y = \sin x$ in the x-axis.)

(b) For the values of x where the graph of $y = \sin x$ lies above the x-axis, we have $|\sin x| = \sin x$, so this part of the graph of $y = |\sin x|$ agrees with the graph of $y = \sin x$. For the values of x where the graph of $y = \sin x$ lies below the x-axis, we have $|\sin x| = -\sin x$, so this part of the graph of $y = |\sin x|$ is obtained by reflecting the corresponding part of the graph of $y = \sin x$ in the x-axis. This gives the following graph.



Solution 4.1

(a) The graph of f is as follows.



From the graph we see that

- ◇ the function f is decreasing and so one-one;
- ◇ the image set of f is the whole of $\mathbb{R}$.

Therefore f has an inverse function f^{-1} with domain $\mathbb{R}$ and image set $\mathbb{R}$, the domain of f. We can find the rule of f^{-1} by solving

$$y = f(x) = 2 - 3x$$

to obtain x in terms of y:

$$y = 2 - 3x, \quad \text{so} \quad x = \tfrac{1}{3}(2 - y).$$

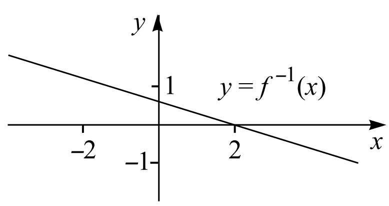
Thus the inverse function is

$$f^{-1}(y) = \tfrac{1}{3}(2 - y),$$

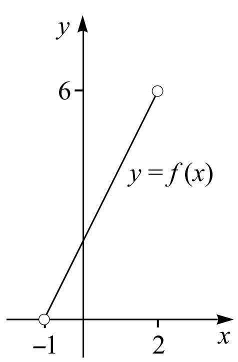
or, in terms of x,

$$f^{-1}(x) = \tfrac{1}{3}(2 - x).$$

The graph of f^{-1} is found by reflecting the graph of f in the 45° line.



(b) The graph of f is as follows.



From the graph we see that

◇ the function f is increasing and so one-one;

◇ the image set of f is $(0, 6)$.

Therefore f has an inverse function f^{-1} with domain $(0, 6)$ and image set $(-1, 2)$, the domain of f. We can find the rule of f^{-1} by solving

$$y = f(x) = 2x + 2$$

to obtain x in terms of y:

$$y = 2x + 2, \quad \text{so} \quad x = \tfrac{1}{2}(y - 2).$$

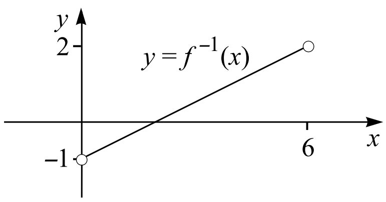
Thus the inverse function is

$$f^{-1}(y) = \tfrac{1}{2}(y - 2) \quad (y \text{ in } (0, 6)),$$

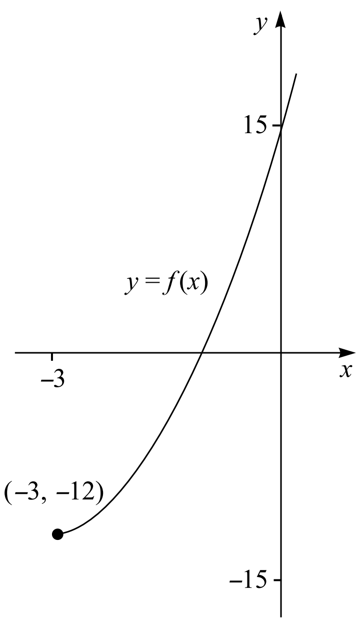
or, in terms of x,

$$f^{-1}(x) = \tfrac{1}{2}(x - 2) \quad (x \text{ in } (0, 6)).$$

The graph of f^{-1} is found by reflecting the graph of f in the 45° line.



(c) The graph of f is as follows (see the solution to Exercise 2.3(a)).



From the graph we see that

◇ the function f is increasing and so one-one;

◇ the image set of f is the interval $[-12, \infty)$.

Therefore f has an inverse function f^{-1} with domain $[-12, \infty)$ and image set $[-3, \infty)$, the domain of f. We can find the rule of f^{-1} by solving

$$y = f(x) = 3x^2 + 18x + 15, \quad \text{where } y \geq -12,$$

to obtain x in terms of y:

$$3x^2 + 18x + 15 - y = 0,$$

so

$$\begin{aligned} x &= \frac{-18 \pm \sqrt{18^2 - 12(15 - y)}}{6} \\ &= \tfrac{1}{6}(-18 \pm \sqrt{144 + 12y}) \\ &= -3 \pm \sqrt{4 + \tfrac{1}{3}y}. \end{aligned}$$

(Alternatively, solve for x using the completed-square form

$$3x^2 + 18x + 15 = 3(x + 3)^2 - 12,$$

obtained in Solution 2.3(a).)

Now, we need to choose the solution of this equation which lies in $[-3, \infty)$, the image set of f^{-1}. Therefore we choose

$$x = -3 + \sqrt{4 + \tfrac{1}{3}y}.$$

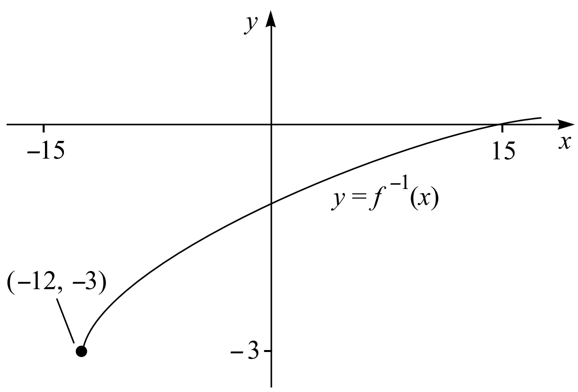
Thus the inverse function is

$$f^{-1}(y) = -3 + \sqrt{4 + \tfrac{1}{3}y} \quad (y \text{ in } [-12, \infty)),$$

or, in terms of x,

$$f^{-1}(x) = -3 + \sqrt{4 + \tfrac{1}{3}x} \quad (x \text{ in } [-12, \infty)).$$

The graph of f^{-1} is found by reflecting the graph of f in the 45° line.



Solution 4.2

(a) From the graph, the function f (with domain $[0, 15]$) appears to be neither increasing nor decreasing. It is also many-one.
From the graph, the function g (with domain $[-10, -5]$) appears to be to be increasing and one-one.
From the graph, the function h (with domain $[-20, -15]$) appears to be decreasing and one-one.

(b) An inverse function exists whenever the original function is one-one on its domain. Thus each of g and h has an inverse function.

(c) The depth of the pool increases fairly steadily from 1 metre at the end where $x = 0$ up to about 2.5 metres where $x \simeq 12$ and then decreases to 2 metres at the far end of the pool where $x = 15$. The pool is deepest when $x \simeq 12$ and shallowest at the end where $x = 0$.

Solution 4.3

(a) This identity is correct. It is true in general that $\tan(\arctan(x)) = x$, for all real numbers x.

(b) This identity is not correct. The image set of arctan is the open interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and -2 does not lie in this interval, as $-2 < -\frac{1}{2}\pi \simeq -1.57$.

(In fact, $\arctan(\tan(-2)) = -2 + \pi$.)

(c) This identity is not correct. The left-hand side equals the angle θ with $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$ for which $\sin\theta = \sin(-\frac{1}{5}\pi)$, namely the angle $-\frac{1}{5}\pi$. But the right-hand side equals the angle θ with $0 \le \theta \le \pi$ for which $\cos\theta = \cos(-\frac{1}{5}\pi)$, namely the angle $\frac{1}{5}\pi$.

(d) This identity is correct. We have $\tan(-\frac{3}{8}\pi) = \tan(\frac{5}{8}\pi)$ (using the result $\tan(x+\pi) = \tan x$ from Section 3 of Chapter A3). Thus both sides of the identity involve taking the arctan of the same number, so they are equal.

(e) This identity is not correct. The expression $\arcsin(\pi)$ is not defined, as π does not lie in the domain $[-1, 1]$ of the function arcsine. (Another explanation is that there is no angle θ for which $\sin\theta = \pi$, as π does not lie in the image set of sine.)

Solution 4.4

(a) $\log_3 1 = 0$

(b) $\log_5(\frac{1}{25}) = \log_5(5^{-2}) = -2$

(c) $\ln(e^{3.1}) = \log_e(e^{3.1}) = 3.1$

(d) $\log_{10}(\sqrt{1000}) = \log_{10}(10^{3/2}) = \frac{3}{2}$

(e) $e^{\ln 4} = e^{\log_e 4} = 4$

(f) $2^{3\log_2 5} = (2^{\log_2 5})^3 = 5^3 = 125$

Solution 4.5

(a) Applying the function ln to both sides of the equation $3^x = 70$, we obtain

$$x\ln(3) = \ln(70),$$

so

$$x = \frac{\ln(70)}{\ln(3)} \simeq 3.87.$$

(Given that a scientific calculator has a button for $\log_{10}$ as well as one for ln, we could equally have applied $\log_{10}$ to each side of the equation, to obtain

$$x = \frac{\log_{10}(70)}{\log_{10}(3)} \simeq 3.87.)$$

(b) If you recognise that $81 = 9^2 = 3^4$, then you can immediately solve the equation to obtain $x = 4$. The approach in part (a) would give

$$x = \frac{\ln(81)}{\ln(3)} = 4.$$

Solution 4.6

If $x = \log_3 6$, then $6 = 3^x$. To solve this equation for x we apply ln to both sides:

$$\ln(3^x) = \ln 6, \quad \text{so} \quad x\ln 3 = \ln 6,$$

giving

$$\log_3 6 = x = \frac{\ln 6}{\ln 3} \simeq 1.63.$$

Solution 4.7

(a)
$$\begin{aligned} \ln(\tfrac{1}{2}xy^3) &= \ln(\tfrac{1}{2}) + \ln x + \ln(y^3) \quad \text{(by property (b)(i))} \\ &= \ln(\tfrac{1}{2}) + \ln x + 3\ln y \quad \text{(by property (c))} \\ &= -\ln 2 + \ln x + 3\ln y \quad \text{(by property (b)(ii))} \\ &= \ln x + 3\ln y - \ln 2 \quad \text{(re-arranging terms).} \end{aligned}$$

(b)
$$\begin{aligned} \ln\left(\frac{3x^2-12}{e^{2x}}\right) &= \ln(3x^2-12) - \ln(e^{2x}) \\ &= \ln(3(x^2-4)) - 2x \\ &= \ln\big(3(x-2)(x+2)\big) - 2x \\ &= \ln 3 + \ln(x-2) + \ln(x+2) - 2x. \end{aligned}$$

Solution 4.8

At the start of year 1, the rabbit population is

$$P_1 = 1400(1.3)^0 + 4200 = 5600.$$

Thus we need to solve the equation

$$P_n = 4 \times 5600 = 22\,400;$$

that is,

$$1400(1.3)^{n-1} + 4200 = 22\,400,$$

or

$$(1.3)^{n-1} = \frac{22\,400 - 4200}{1400} = 13.$$

Applying ln to both sides and rearranging, we obtain

$$n - 1 = \frac{\ln(13)}{\ln(1.3)} \simeq 9.78, \quad \text{so} \quad n \simeq 10.78.$$

Thus P_n will reach 22 400 during year 10.

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